RECONSTRUCTIVE TOMOGRAPHY WITH WEAK OPTICAL ANISOTROPY

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Integral photoelasticity [1] is concerned with the problem of determining the stressed state of a body with the help of transillumination. In this sense integral photosensitivity can be thought of as optical tomography of a tensor field [2].

In the presence of weak optical anisotropy [2] it is possible to measure, by means of transillumination along a ray, two integrals of the components of the stress tensor σ_{ij}. There arises the problem of using the maximum information contained in these ray integrals and finding the stressed state of the body from them. A partial answer to these questions and the corresponding bibliography are contained in [1-4].

From the technical standpoint the transillumination is most conveniently conducted in parallel planes (z = const). It is precisely for such measurements that a method for determining σ_{zz} is given, and for a linearly deformable body a class of stressed states which make it possible to calculate all components of the stress tensor from $\sigma_{zz}(x, y, z)$ is identified.

1. We introduce an orthogonal coordinate system in a cylindrical body whose side surface is free. As the z axis we take the axis passing through the center of gravity of the cross section S and as the x and y axis we take the inertial axes of the cross section.

The components of the stress tensor satisfy the equations of equilibrium

$$\sigma_{ij,i} = 0 \ (i, j = x, y, z) \tag{1.1}$$

and the boundary conditions on the contour Γ — the free side surface —

$$\sigma_{xi}n_x + \sigma_{yi}n_y = 0. \tag{1.2}$$

We shall write the condition of equilibrium of an element S(m, θ) of thickness Δz in the direction of the z axis with the help of the ray integral (see Fig. 1):

$$H(m, \theta, z) = \int \sigma_{iz} m_i dl = -\int \sigma_{xz} dy - \sigma_{yz} dx. \tag{1.3}$$

Here m_i are the components of the unit vector normal to the ray ℓ; $m_x = \cos\theta$; $m_y = \sin\theta$; and, m is the distance from the origin of the coordinates to the straight line ℓ. The difference of the forces on the top and bottom surfaces of the element is balanced by the tangential force on the side surface ($z_0 \in [z, z + \Delta z]$)

$$\Delta z H(m, \theta, z) = \int_m^{m_1} \int [\sigma_{zz}(m', z + \Delta z) - \sigma_{zz}(m', z)\, dl\, dm'].$$

Dividing both sides by Δz and passing to the limit Δz → 0, we put the condition of equilibrium into the differential form

$$H(m, \theta, z) = \int_m^{m_1} \int \frac{\partial}{\partial z} \sigma_{zz}(m', l, z)\, dl\, dm'$$

and then, differentiating with respect to m, into the form of a Radon transformation

$$\frac{\partial}{\partial m} H(m, \theta, z) = -\int \frac{\partial}{\partial z} \sigma_{zz}(m, l, z)\, dl.$$

Thus finding $\frac{\partial}{\partial z}\sigma_{zz}$ from the ray integral H reduces to the standard procedure of inverting the Radon transform [5].

We represent σ_{zx} and σ_{yz} as two-dimensional vectors in a plane in the form of a sum of the two-dimensional gradient and the curl with potentials F and G:

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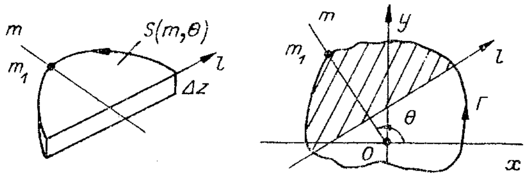

Fig. 1

$$\sigma_{xz} = -\frac{\partial}{\partial x}F + \frac{\partial}{\partial y}G, \quad \sigma_{yz} = -\frac{\partial}{\partial y}F - \frac{\partial}{\partial x}G. \tag{1.4}$$

We introduce the coordinate t, measured counterclockwise along the contour Γ. The boundary conditions (1.2) for σ_{iz}, taking into account Eq. (1.4), are transformed as follows: $\frac{\partial}{\partial t}G - \frac{\partial}{\partial n}F = 0$ on Γ. We substitute Eq. (1.4) into the equation of equilibrium $\sigma_{iz,i} = 0$ and write F as a sum $F = F_+ + F_0$, so that

$$\Delta_+F = \Delta_+F_+ = \frac{\partial}{\partial z}\sigma_{zz}, \quad \frac{\partial}{\partial n}F_+ = 0 \text{ on } \Gamma, \quad \Delta_+F_0 = 0,$$
$$\frac{\partial}{\partial n}F_0 = \frac{\partial}{\partial t}G \text{ on } \Gamma \left(\Delta_+F = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F\right). \tag{1.5}$$

If there is no normal rotation (G = 0), then the function F_+ and therefore σ_{xz} and σ_{yz} are determined from the solution of the Poisson equation (1.5). Otherwise the solution F_+ determines only partially the potential component of the stresses.

It is interesting to note that H can be quite easily expressed in terms of F_+. Indeed, we represent the integral H as a contour integral, closing the contour by a segment of arc of the curve Γ. Substituting Eq. (1.4) in to Eq. (1.3), taking into account what we have said above and the relations (1.5), reduces H to the form

$$H(m, \theta, z) = \int \frac{\partial}{\partial x}F_+\,dy - \frac{\partial}{\partial y}F_+\,dx = \int \frac{\partial}{\partial m}F_+\,dl.$$

The latter expression can also be used as the basic expression for constructing computational algorithm.

2. We shall now study the reduction of a different ray integral (A) [2]: $A(m, \theta, z) = \int m_i m_j \sigma_{ij} - \sigma_{zz}\,dl$. We shall first derive a formula which relates P(m, θ) – the projection of the principal vector of the force of the cross section S(m, θ) on the direction m (see Fig. 1) – with the integrals A and H:

$$P(m, \theta) = \int\int \sigma_{iz} m_i\,dx\,dy. \tag{2.1}$$

For this we integrate over S(m, θ) the equations of equilibrium contracted with the vector m_i:

$$\int\int m_i \frac{\partial}{\partial z}\sigma_{iz}\,dx\,dy = -\int\int m_i\left(\frac{\partial}{\partial x}\sigma_{ix} + \frac{\partial}{\partial y}\sigma_{iy}\right)dx\,dy = -\int m_i m_j \sigma_{ij}\,dl = \frac{\partial}{\partial z}P(m, \theta, z). \tag{2.2}$$

In Eq. (2.2) the fact that the side surface is free of any loads was taken into account.

We shall perform the integration over the variables ℓ and m in Eq. (2.1) (m_1 is the value of m on the contour Γ):

$$P(m, \theta, z) = \int\int_{S(m,\theta)} \sigma_{iz} m_i\,dl\,dm = \int_{m_1}^{m} H(m_0, \theta, z)\,dm_0. \tag{2.3}$$

Substituting Eq. (2.3) transforms the expression (2.2) into the expression

$$\int \sigma_{zz}\,dl = -\int_{m_1}^{m} \frac{\partial}{\partial z}H(m_0, \theta, z)\,dm_0 - A(m, \theta, z). \tag{2.4}$$

Thus according to Eq. (2.4) the construction of σ_{zz} from A and H reduces to a well-developed procedure – inversion of the Radon transform [5]. The formula (2.2) and, correspondingly, Eq. (2.4) have a simple physical meaning: They are the conditions of equilibrium of an infinitely thin elements S(m, θ) (see Fig. 1).

It follows from everything said above that only the components σ_{zz} and the derivative with respect to z are determined from the results of transillumination in parallel planes z = const. In what follows a particular formulation of the problem, which makes it possible to find the remaining components from these values, is analyzed.

3. We shall assume in addition that Hooke's law is satisfied and there is no normal rotation (G = 0). Therefore σ_{xz} and σ_{yz} are determined in the volume in addition to σ_{zz}. Let us assume that these components are known in two sections z_1 and z_2. Then the loads on the surface of the volume singled out are completely given and the stresses inside the volume are found from the solution of the second basic boundary-value problem of the theory of elasticity [6]. Reducing $\Delta z = (z_2 - z_1)$ (the height of the part that was singled out) we reduce the problem to a two-dimensional problem, i.e., we determine all stresses layer by layer. This substantially simplifies the calculations. The problem is formulated in terms of stresses; the conditions (1.2) are satsified on the side surface. We assume that σ_{zz} and correspondingly F are known from the solution of Poisson's equation (1.5).

To find the invariant $\sigma = \Sigma\sigma_{ii}$ we substitute Eq. (1.4) into the two compatibility equations

$$(1+\nu)\Delta\sigma_{iz} + \sigma_{,iz} = 0,\ i = x, y,$$

and transform these equations as follows:

$$\frac{\partial^2}{\partial z\,\partial i}\left[\frac{\partial}{\partial z}F + \sigma_{zz} - \frac{1}{1+\nu}\sigma\right] = 0,\ i = x, y. \tag{3.1}$$

Here the equality, taking into account Eq. (1.5), we employed:

$$\Delta F = \frac{\partial^2 F}{\partial z^2} + \Delta_+ F = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}F + \sigma_{zz}\right].$$

The value of the invariant σ as the general solution of the equations of compatibility (3.1) can be represented by a sum of particular and general solutions for $\sigma_{zz} = F = 0$. We obtain the particular solution by equating the expression in brackets to zero, and by taking into account the third equation of compatibility $\Delta\sigma_{zz} + \sigma_{,zz} = 0$ we reduce the additional terms of the general solution to the form $I_0(x, y) = I_1 z$.

In the case of two-dimensional strain $\sigma_{zz} = \sigma_{zz}^{+}(z, y)$ depends only on x and y, the potential F = 0, and the expression of the invariant transforms into the well-known expression [6] and thereby determines $I_0(x, y)$; in addition, $I_1 = 0$.

Finally the solution

$$\sigma = (1+\nu)\left[\sigma_{zz} - \sigma_{zz}^{+} + \frac{\partial F}{\partial z}\right] + \frac{1}{\nu}\left[(1+\nu)\sigma_{zz}^{+} - (a_0 + a_1 x + a_2 y)\right]$$

is expressed in terms of F – the value vanishing for $\sigma_{zz} = \sigma_{zz}^{+}$; a_0, a_1, and a_2 are constants, which determine the homogeneous strain with respect to z and the "pure" bending.

We find the components σ_{xx} and σ_{yy} from the two equations of equilibrium (1.1) with j = x, y. Substituting into them $\sigma_{xx} = \sigma - \sigma_{yy} - \sigma_{zz}$ and eliminating σ_{xy} we obtain an equation for σ_{yy}:

$$\Delta_+\sigma_{yy} = \frac{\partial^2}{\partial x^2}\left[\sigma - \sigma_{zz} - \frac{\partial F}{\partial z}\right] + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial z}\right).$$

The boundary conditions (1.2) for the component σ_{yy} is reduced by such a substitution to the equation $\sigma_{yy} = (\sigma - \sigma_{zz})n_x^2$.

The solution of Eqs. (1.1) for σ_{xy} can be represented by the curvilinear integral

$$\sigma_{xy} = -\int \frac{\partial}{\partial x}\left[\sigma_{xx} - \frac{\partial F}{\partial z}\right]dy + \frac{\partial}{\partial y}\left[\sigma_{yy} - \frac{\partial F}{\partial z}\right]dx + \sigma_{xy}^0. \tag{3.2}$$

Here σ_{xy}^0 is chosen on the contour and satisfies the boundary conditions (1.1). We shall choose for the initial point of integration in Eq. (3.2) a point on the contour Γ, where $n_y = 0$, so that $\sigma_{xy}^0 = 0$.

It is obvious from the construction that the stress in the plane is actually given by the value of $\sigma_{zz}(x, y)$ in it and by the first two derivatives of this functions with respect to z.

The first derivative of σ_{zz} with respect to z determines σ_{zx} and σ_{zy}, and the combination of σ_{zz} and its second derivative with respect to z determines the remaining components of the stress. These two functions must satisfy the condition that the integral (3.2) must vanish on the contour Γ. This condition is a consequence of the assumption that the stressed state does not contain a normal rotation (G = 0) and can be used in the algorithm in order to reduce the errors of the starting measurements.

Problems formulated in this manner have thus far been solved only for the axisymmetric stressed state [4, 7]. The stressed state not containing normal rotation is significantly more extensive and contains an axisymmetric state as a particular case.

In conclusion it is my pleasant duty to thank Kh. K. Aben for proposing the subject and also for his constant well-meaning interest in this work.

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DYNAMIC DUCTILITY PEAK WITH HIGH-VELOCITY FAILURE OF METAL SHELLS

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An explanation of the dynamic ductility peak [1] is given. It is shown that this effect is connected with a sharp deterioration in metal ductility properties with a strain rate of $\dot{\varepsilon} \sim 10^5$ sec^{-1}.

The failure of cylindrical metal shells expanding under the action of detonation products with high strain rates of $\dot{\varepsilon} > 10^4$ sec^{-1} was studied in [1-4]. Here it was detected that high-velocity failure exhibits a number of features which relate to existence of a scale effect and a dynamic ductility peak.

An explanation of these features will be sought within the scope of describing failure as a two-stage process [2]. The first stage consists of damage accumulation with plastic flow. In the second stage by crack propagation there is separation of the shell parts due to stored elastic energy.

We divide the process of damage accumulation into two stages. We assume that in the first stage there is accumulation of point defects, and in the second there is growth of pores which are the result of merging of point defects. Similar to [5] we shall assume that defects arise with unconservative movement of steps which form from the intersection of edge and screw dislocations. Then from [5] it follows that the concentration of defects $c_d = f(\varepsilon)$. Since occurrence of pores occurs with some critical concentration of them c_d^*, the material should experience some strain ε_0 prior to pore growth commencing.

In the second stage pore growth is determined by the viscosity and inertial properties of the material. This assumption is correct with high strain rates. According to [6] the equation for pore radius has the form

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